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Dynamics of a mean spherical model with competing interactions

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Abstract
The Langevin dynamics of a d-dimensional mean spherical model with competing interactions along $m \leqslant d$ directions of a hypercubic lattice is analysed. After a quench at high temperatures, the dynamical behaviour is characterized by two distinct time scales associated with stationary and ageing regimes. The asymptotic expressions for the autocorrelation and response functions, in supercritical, critical and subcritical cases, were calculated. Ageing effects, which are known to be present in the ferromagnetic version of this model system, are not affected by the introduction of competing interactions.

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1. Introduction

Non-equilibrium phenomena, such as ageing and violations of the fluctuation–dissipation theorem (FDT), have been attracting the attention of many investigators. A number of dynamical calculations for disordered as well as uniform magnetic model systems point out the occurrence of ageing and violations of the FDT in a time evolution from a quench at high temperatures [1–3]. Since there are no general principles to understand and classify these dynamical phenomena, it has been valuable to analyse the dynamical behaviour of simple, analytically tractable, model systems. In the present work, a detailed investigation of the Langevin dynamics of an analytically tractable d-dimensional mean spherical model with competing interactions is reported.

Spin models with competing interactions are known to display a rich phase diagram, with multicritical points and modulated phases. In terms of the temperature T and of a parameter p gauging the strength of the competing interactions, the phase diagram of the axial next-nearest-neighbour Ising (or ANNNI [4, 5]) model displays a Lifshitz point, at the meeting of paramagnetic–ferromagnetic and paramagnetic-modulated critical lines, and an impressive sequence of modulated structures at low temperatures. The thermodynamic behaviour of a spherical version of an Ising model with competing interactions has been originally analysed by Kalok and Obermair [6]. A spherical analogue of the ANNNI model, with the characterization of a Lifshitz point and the existence of ordered ferromagnetic and helical phases, has been introduced by Hornreich and co-workers [7]. The field behaviour of this spherical analogue of the ANNNI model has been investigated by Yokoi and co-workers [8]. With the exception of a few numerical works, as the analysis of an Ising model with both ferromagnetic and antiferromagnetic dipolar interactions in order to account for the behaviour of ultrathin magnetic films [9], we are not aware of analytical investigations of the dynamics of statistical models with competing interactions.

In a recent paper, Godrèche and Luck [2] reported a detailed analytical treatment of the Langevin dynamics of a d-dimensional ferromagnetic mean spherical model. The present work may be regarded as an extension of this analysis for a mean-spherical model with competing interactions. The particular results of Godrèche and Luck are recovered.

The layout of this paper is as follows. The spherical model with competing interactions is introduced in section 2. The Langevin dynamics, with the inclusion of a time-dependent Lagrange multiplier for implementing the spherical constraint, is analysed in section 3, but the mathematical details of this analysis are left for the appendix. In section 4, some comments are made and the main conclusions are presented.

2. Definition of the model

The grand canonical partition function,

$$\Xi_N(\beta,\mu) = \int \exp\left[-\beta H(\{S_x\}) - \beta\mu \sum_{x\in\Lambda_N} S_x^2\right] \prod_{x\in\Lambda_N} \mathrm{d}S_x, \tag{1}$$

subjected to a spherical constraint,

$$\left\langle \sum_{x\in\Lambda_N} S_x^2 \right\rangle = -\frac{1}{\beta}\frac{\partial}{\partial\mu} \ln \Xi_N(\beta,\mu) = N, \tag{2}$$

is the trademark of a mean spherical model. The spin variables $S_x \in \mathbb{R}$ are continuous, β is the (inverse) temperature, μ is a Lagrange multiplier that canonically ensures the spherical constraint and $\Lambda_N = \{-L, -L+1, \ldots, L, L+1\}^d$ is a hypercubic lattice with N sites. The Hamilton function is given by

$$H(\{S_x\}) = -\frac{1}{2} \sum_{x,x'\in\Lambda_N} J_{x,x'} S_x S_{x'}, \tag{3}$$

where

$$J_{x,x'} = \begin{cases} RJ, & x - x' = \pm e_i \quad i \in \{1,\ldots,m\} \\ SJ, & x - x' = \pm 2e_i \quad i \in \{1,\ldots,m\} \\ J, & x - x' = \pm e_i \quad i \in \{m+1,\ldots,d\} \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

This exchange integral describes the whole features of the model. There are nearest- and next-nearest-neighbour interactions along m ($\leqslant d$) out of the d directions of the hypercubic lattice; along the remaining $d-m$ directions, there are only ferromagnetic, $J > 0$, nearest-neighbour interactions. We assume periodic boundary conditions along each direction ($S_{L+1} = S_{-L}$). Parameters R and S are free, but the scenario of competition takes place for $S < 0$. In the

Table 1. Lower and upper critical dimensions.

$m \neq 0, d-m \neq 0$	Case 1	$\lvert R\rvert + 4S \neq 0$	$d_c = 2$	$\overline{d} = 4$
	Case 2	$\lvert R\rvert + 4S = 0$	$d_c = \frac{2}{2-r}$	$\overline{d} = \frac{4}{2-r}$
$m \neq 0, d-m = 0$	Case 3	$\lvert R\rvert + 4S \neq 0$	$d_c = 2$	$\overline{d} = 4$
	Case 4	$\lvert R\rvert + 4S = 0$	$d_c = 4$	$\overline{d} = 8$
$m = 0, d-m \neq 0$	Case 5	$\times$	$d_c = 2$	$\overline{d} = 4$

particular (ferromagnetic) case analysed by Godrèche and Luck, $m = 0$. The simple spherical analogue of the ANNNI model is given by $m = 1$ (with the parameter $p = -S/R$ gauging the strength of the competition).

The partition function can be obtained by standard procedures [10]. In the thermodynamic limit, the spherical constraint leads to the relation

$$\beta(\mu) = \int_{[-\pi,\pi]^d} \frac{\mathrm{d}^d k}{2(2\pi)^d} \frac{1}{\mu - \frac{1}{2}\hat{J}(k)}, \tag{5}$$

where β is written in terms of the Lagrange multiplier μ, and

$$\mu \geqslant \mu_c := \tfrac{1}{2}\hat{J}(k_c) = \tfrac{1}{2} \sup_{k\in[-\pi,\pi]^d} \{\hat{J}(k)\}, \tag{6}$$

where

$$\hat{J}(k) = 2J\left[R\sum_{i=1}^{m}\cos k_i + S\sum_{i=1}^{m}\cos(2k_i) + \sum_{i=m+1}^{d}\cos k_i\right] \tag{7}$$

is the Fourier transform of the exchange integral. The critical wave vector k_c comes from equation (6). Thus, one can write

$$k_c = (\underbrace{q_c,\ldots,q_c}_{m}, \underbrace{0,\ldots,0}_{d-m}), \tag{8}$$

where this point in the first Brillouin zone satisfies condition (6), and is determined by the parameters R and S. It is easy to see that

$$q_c = \begin{cases} 0, & R > 0 \text{ and } S > -R/4 \\ \pi, & R < 0 \text{ and } S > -R/4 \\ \pm\phi, & S < -|R|/4, \end{cases} \tag{9}$$

where $\phi := \arccos\left(-\frac{R}{4S}\right)$.

The sum rule (5), which defines the critical temperature $\beta(\mu_c)$, also leads to the lower and upper critical dimensions, d_c and $\overline{d}$, which are listed in table 1, in terms of d, m and the ratio $r = m/d$. Note that it is convenient to introduce and analyse five different cases. Also note that these ingredients will be sufficient for characterizing the asymptotic dynamical behaviour.

3. The Langevin dynamics

The dynamics is assumed to be governed by the Langevin equation,

$$\frac{\partial S_x(t)}{\partial t} = -\frac{\delta}{\delta S_x(t)}\left\{H[S_x](t) + \mu(t)\sum_{x\in\Lambda} S_x^2(t)\right\} + \xi_x(t), \tag{10}$$

where $\{\xi_x(t)\}$ is a set of random variables such that

$$\langle \xi_x(t) \rangle = 0 \qquad \text{and} \qquad \langle \xi_x(t)\xi_{x'}(t') \rangle = 2T\delta_{x,x'}\delta(t-t'). \tag{11}$$

In contrast to the static case, the Lagrange multiplier μ is now a function of time, ensuring the spherical constraint at each time t.

In this work, the calculations are limited to the long-time behaviour of autocorrelations and response functions. The analysis is restricted to the asymptotic expansions (for large t') of the autocorrelation,

$$C(t,t') := \frac{1}{N}\sum_{x\in\Lambda_N} \langle S_x(t)S_x(t') \rangle = \frac{1}{N}\sum_{k\in\hat{\Lambda}_N} C_k(t,t'), \tag{12}$$

with $t > t'$, where $C_k(t,t') = \langle S_k(t)S_{-k}(t') \rangle$ is a two-time correlation in the Fourier space $\hat{\Lambda}$, and the response function:

$$R(t,t') := \frac{1}{N}\sum_{x\in\Lambda_N} \left.\frac{\delta S_x(t)}{\delta h_x(t')}\right|_{h\downarrow 0} = \frac{1}{N}\sum_{k\in\hat{\Lambda}_N} R_k(t,t'), \tag{13}$$

where $R_k(t,t') = \delta\langle S_x(t)\rangle/\delta h_k(t')$, and h is just a small perturbation. According to the fluctuation–dissipation theorem, in a stationary regime these functions are related by the expression

$$X(t,t') = \frac{TR(t,t')}{\partial_{t'}C(t,t')} = 1. \tag{14}$$

If $X(t,t') \neq 1$ the theorem is violated, which suggests the introduction of an effective temperature $T/X(t,t')$, larger than the heat-bath temperature T, and which is supposed to gauge a non-stationary behaviour of the system.

In order to calculate the two-time functions, one may first define the functional

$$\psi[\mu](t) := \exp\left[4\int_0^t \mu(t')\,\mathrm{d}t'\right], \tag{15}$$

which will be denoted by $\psi(t)$. By solving the differential equation (10) and using the definition of $C_k(t,t')$, one can show that

$$C_k(t,t') = \frac{1}{\sqrt{\psi(t)\psi(t')}}\left\{C_k(0,0)\exp[\hat{J}(k)(t+t')] + 2T\int_0^{t'} \exp[\hat{J}(k)(t+t'-2t'')]\psi(t'')\,\mathrm{d}t''\right\}, \tag{16}$$

where $C_k(0,0)$ is the initial condition. For a quench from a totally disordered state, at an effectively infinite temperature, one should take $C_k(0,0) = 1$.

The autocorrelation is obtained from the spherical constraint $C(t,t) = 1$ (see (12) and (2)), which implies, in the thermodynamics limit, and for $t \geqslant 0$,

$$\psi(t) = f(t) + 2T\int_0^t f(t-t')\psi(t')\,\mathrm{d}t', \tag{17}$$

with

$$f(t) := \int_{[-\pi,\pi]^d} \frac{\mathrm{d}^d k}{(2\pi)^d}\,\mathrm{e}^{2\hat{J}(k)t} = [I_0(4Jt)]^{d-m}\left[\frac{1}{\pi}\int_0^\pi \mathrm{e}^{4Jtg(k)}\,\mathrm{d}k\right]^m, \tag{18}$$

where $I_0(x)$ is the modified Bessel function of order zero, and

$$g(k) := R\cos k + S\cos(2k) \tag{19}$$

corresponds to the portion of the exchange integral responsible for the competition (if $S < 0$).

Table 2. K_p and γ_p.

	K_p	γ_p
Case 1	$2^m(8\pi J)^{-\frac{d}{2}}\left\|g^{(2)}(q_c)\right\|^{-\frac{m}{2}}$	$\frac{d}{2}$
Case 2	$(8\pi J)^{-\frac{2d-m}{4}}\left(\frac{48}{\pi^3}\right)^{\frac{m}{4}}\Gamma\left(\frac{5}{4}\right)^m\left\|g^{(4)}(q_c)\right\|^{-\frac{m}{4}}$	$\frac{2d-m}{4}$
Case 3	$2^d(8\pi J)^{-\frac{d}{2}}\left\|g^{(2)}(q_c)\right\|^{-\frac{d}{2}}$	$\frac{d}{2}$
Case 4	$(8\pi J)^{-\frac{d}{4}}\left(\frac{48}{\pi^3}\right)^{\frac{d}{4}}\Gamma\left(\frac{5}{4}\right)^d\left\|g^{(4)}(q_c)\right\|^{-\frac{d}{4}}$	$\frac{d}{4}$
Case 5	$(8\pi J)^{-\frac{d}{2}}$	$\frac{d}{2}$

The convolution product in equation (17) suggests a solution by Laplace transform, which yields

$$\psi(t) = \frac{1}{2\pi\mathrm{i}}\int_{\sigma-\mathrm{i}\infty}^{\sigma+\mathrm{i}\infty}\mathrm{d}s\,\mathrm{e}^{st}\frac{\mathcal{L}[f](s)}{1-2T\mathcal{L}[f](s)}, \tag{20}$$

where σ is larger than any poles of the integrand. The problem is now to determine ψ, since the autocorrelation,

$$C(t,t') = \frac{1}{\sqrt{\psi(t)\psi(t')}}\left[f\left(\frac{t+t'}{2}\right)+2T\int_0^{t'}\mathrm{d}y\,f\left(\frac{t+t'}{2}-y\right)\psi(y)\right], \tag{21}$$

and the response function,

$$R(t,t') = f\left(\frac{t-t'}{2}\right)\sqrt{\frac{\psi(t')}{\psi(t)}}, \tag{22}$$

can be both written in terms of ψ.

The asymptotic behaviour of f is

$$f(t) \sim K_p\frac{\mathrm{e}^{2\hat{J}(k_c)t}}{t^{\gamma_p}}, \tag{23}$$

where p labels the set $p = \{R, S, m, d\}$. As it is fully discussed in the appendix, the expressions for K_p and γ_p, which are listed in table 2, depend on the four parameters $\{R, S, m, d\}$.

The behaviour of ψ for large times demands the asymptotic expansion of $\mathcal{L}[f]$ for small values of $\epsilon := s - 2\hat{J}(k_c) > 0$. With the same notation, $r = m/d$, and $p = \{R, S, m, d\}$, one can calculate the asymptotic expression

$$\mathcal{L}[f](s) \sim \begin{cases} \frac{F_p g_p}{\epsilon^{-\alpha_p}} & 0 < d < d_c \\ F_p\,(-\ln\epsilon) & d = d_c \\ A_1 - F_p|g_p|\epsilon^{\alpha_p} & d_c < d < \overline{d} \\ A_1 - F_p\,(-\epsilon\ln\epsilon) & d = \overline{d} \\ A_1 - F_p\epsilon & d > \overline{d}, \end{cases} \tag{24}$$

with the coefficients given in table 3. Note that $g^{(n)}$ is the nth derivative of g, given by equation (19), which is different from g_p. Also, note that $\alpha_p = \gamma_p - 1$.

The next step is the determination of the asymptotic behaviour of ψ. The calculations are separated in three parts, each of them corresponding to a different temperature regime.

Table 3. F_p, g_p and α_p.

Case	F_p	d
Case 1	$\times$	$d < 2$
	$2^m(8\pi J)^{-1}\|g^{(2)}(q_c)\|^{-\frac{m}{2}}$	$d = 2$
$\alpha_p = \frac{d-2}{2}$	$2^m(8\pi J)^{-\frac{d}{2}}\|g^{(2)}(q_c)\|^{-\frac{m}{2}}$	$2 < d < 4$
	$2^m(8\pi J)^{-2}\|g^{(2)}(q_c)\|^{-\frac{m}{2}}$	$d = 4$
$g_p = \Gamma\left(\frac{2-d}{2}\right)$	A_2	$d > 4$
Case 2	$(8\pi J)^{-\frac{2d-m}{4}}\left(\frac{48}{\pi^3}\right)^{\frac{m}{4}}\Gamma\left(\frac{5}{4}\right)^m\|g^{(4)}(q_c)\|^{-\frac{m}{4}}$	$d < \frac{4}{2-r}$
	$(8\pi J)^{-1}\left(\frac{48}{\pi^3}\right)^{\frac{m}{4}}\Gamma\left(\frac{5}{4}\right)^m\|g^{(4)}(q_c)\|^{-\frac{m}{4}}$	$d = \frac{4}{2-r}$
$\alpha_p = \frac{2d-m-4}{4}$	$(8\pi J)^{-\frac{2d-m}{4}}\left(\frac{48}{\pi^3}\right)^{\frac{m}{4}}\Gamma\left(\frac{5}{4}\right)^m\|g^{(4)}(q_c)\|^{-\frac{m}{4}}$	$\frac{4}{2-r} < d < \frac{8}{2-r}$
	$(8\pi J)^{-2}\left(\frac{48}{\pi^3}\right)^{\frac{m}{4}}\Gamma\left(\frac{5}{4}\right)^m\|g^{(4)}(q_c)\|^{-\frac{m}{4}}$	$d = \frac{8}{2-r}$
$g_p = \Gamma\left(\frac{4-2d+m}{4}\right)$	A_2	$d > \frac{8}{2-r}$
Case 3	$2^d(8\pi J)^{-\frac{d}{2}}\|g^{(2)}(q_c)\|^{-\frac{d}{2}}$	$d < 2$
	$2^d(8\pi J)^{-1}\|g^{(2)}(q_c)\|^{-1}$	$d = 2$
$\alpha_p = \frac{d-2}{2}$	$2^d(8\pi J)^{-\frac{d}{2}}\|g^{(2)}(q_c)\|^{-\frac{d}{2}}$	$2 < d < 4$
	$2^d(8\pi J)^{-2}\|g^{(2)}(q_c)\|^{-2}$	$d = 4$
$g_p = \Gamma\left(\frac{2-d}{2}\right)$	A_2	$d > 4$
Case 4	$(8\pi J)^{-\frac{d}{4}}\left(\frac{48}{\pi^3}\right)^{\frac{d}{4}}\Gamma\left(\frac{5}{4}\right)^d\|g^{(4)}(q_c)\|^{-\frac{d}{4}}$	$d < 4$
	$(8\pi J)^{-1}\left(\frac{48}{\pi^3}\right)\Gamma\left(\frac{5}{4}\right)^4\|g^{(4)}(q_c)\|^{-1}$	$d = 4$
$\alpha_p = \frac{d-4}{4}$	$(8\pi J)^{-\frac{d}{4}}\left(\frac{48}{\pi^3}\right)^{\frac{d}{4}}\Gamma\left(\frac{5}{4}\right)^d\|g^{(4)}(q_c)\|^{-\frac{d}{4}}$	$4 < d < 8$
	$(8\pi J)^{-2}\left(\frac{48}{\pi^3}\right)^2\Gamma\left(\frac{5}{4}\right)^8\|g^{(4)}(q_c)\|^{-2}$	$d = 8$
$g_p = \Gamma\left(\frac{4-d}{4}\right)$	A_2	$d > 8$
Case 5	$(8\pi J)^{-\frac{d}{2}}$	$d < 2$
	$(8\pi J)^{-1}$	$d = 2$
$\alpha_p = \frac{d-2}{2}$	$(8\pi J)^{-\frac{d}{2}}$	$2 < d < 4$
	$(8\pi J)^{-2}$	$d = 4$
$g_p = \Gamma\left(\frac{2-d}{2}\right)$	A_2	$d > 4$

3.1. Supercritical dynamics

If a system is quenched from a highly disordered state (for instance, the system may have an effectively infinite temperature) to $T > T_c$, the function ψ has an asymptotic exponential behaviour

$$\psi(t) \sim \mathrm{e}^{t/\tau_p}, \tag{25}$$

where τ_p is related to the characteristic time. This behaviour indicates the decay of the system to an equilibrium state in finite time. In this situation the autocorrelation,

$$C(t, t') \sim C(\tau) = T \int_\tau^\infty \mathrm{d}y\, f\left(\frac{y}{2}\right) \mathrm{e}^{-\frac{y}{2\tau_p}}, \tag{26}$$

and the response function,

$$R(t, t') \sim R(\tau) = f\left(\frac{\tau}{2}\right) \mathrm{e}^{-\frac{\tau}{2\tau_p}}, \tag{27}$$

depend on the time difference τ only, and the fluctuation–dissipation theorem is satisfied,

$$X(t, t') \sim 1. \tag{28}$$

3.2. Critical dynamics

In contrast to the other cases, the critical dynamical behaviour depends on the dimension of the system. In the following calculations, it will always be assumed that d is larger than the lower critical dimension d_c, so that $T_c \neq 0$; in other words, the occurrence of a phase transition is assumed.

The asymptotic behaviour of ψ is given by

$$\psi(t) \sim \begin{cases} \frac{e^{2\hat{J}(k_c)t}}{t^{1-\alpha_p}}, & d_c < d < \overline{d} \\ \frac{e^{2\hat{J}(k_c)t}}{\ln t}, & d = \overline{d} \\ e^{2\hat{J}(k_c)t}, & d > \overline{d}, \end{cases} \tag{29}$$

which is sensitive to the dimension.

Two time scales arise in the analysis of the critical dynamics:
(i) In the stationary regime, $1 \sim \tau \ll t'$, both the autocorrelation,

$$C(t,t') \sim C_{\mathrm{eq},c}(\tau), \tag{30}$$

with

$$C_{\mathrm{eq},c}(\tau) := T_c \int_\tau^\infty \mathrm{d}y\, f\left(\frac{y}{2}\right) e^{-\hat{J}(k_c)y}, \tag{31}$$

and the response function,

$$R(t,t') \sim f\left(\frac{\tau}{2}\right) e^{-\hat{J}(k_c)\tau}, \tag{32}$$

are invariant under time translation. The fluctuation–dissipation theorem is satisfied with $X(t,t') \sim 1$. The choice $\tau \sim 1$ precludes the system to decay from the stationary state, which suggests the occurrence of ageing for larger values of τ.
(ii) For $1 \ll \tau \sim t'$, it is convenient to define

$$x := \frac{t}{t'}. \tag{33}$$

In this regime, the autocorrelation,

$$C(t,t') \sim \begin{cases} \frac{2K_pT_c2^{\gamma_p}}{\gamma_p-1} t'^{1-\gamma_p} \frac{x^{1-\frac{\gamma_p}{2}}(x-1)^{1-\gamma_p}}{x+1}, & d_c < d < \overline{d} \\ \frac{T_cK_p2^{\gamma_p}}{\gamma_p-1} t'^{1-\gamma_p} \sqrt{1+\frac{\ln x}{\ln t'}} \times & \\ \times\left[(x-1)^{1-\gamma_p} - (x+1)^{1-\gamma_p}\right], & d = \overline{d} \\ \frac{2^{\gamma_p}K_pT_c}{\gamma_p-1} t'^{1-\gamma_p}\left[(x-1)^{1-\gamma_p} - (x+1)^{1-\gamma_p}\right], & d > \overline{d}, \end{cases} \tag{34}$$

and the response function,

$$R(t,t') \sim \begin{cases} 2^{\gamma_p}K_p t'^{-\gamma_p} x^{\frac{1-\alpha_p}{2}} (x-1)^{-\gamma_p}, & d_c < d < \overline{d} \\ 2^{\gamma_p}K_p t'^{-\gamma_p} (x-1)^{-\gamma_p} \sqrt{1+\frac{\ln x}{\ln t'}}, & d = \overline{d} \\ 2^{\gamma_p}K_p t'^{-\gamma_p} (x-1)^{-\gamma_p}, & d > \overline{d}, \end{cases} \tag{35}$$

show that the time translation invariance is broken, and ageing effects are observed.

The asymptotic behaviour of the fluctuation–dissipation ratio is calculated from equations (34) and (35), which yield

$$X(t,t') \sim \begin{cases} \frac{(\gamma_p-1)(x+1)^2}{(\gamma_p x+\gamma_p-2)(x+1)-2(x-1)}, & d_c < d < \bar{d} \\ \frac{2(\gamma_p-1)\ln t'}{2(\gamma_p-1)\left[1+\left(\frac{x-1}{x+1}\right)^{\gamma_p}\right]\ln t'-(x-1)\left[1-\left(\frac{x-1}{x+1}\right)^{\gamma_p-1}\right]}, & d = \bar{d} \\ \frac{1}{1+\left(\frac{x-1}{x+1}\right)^{\gamma_p}}, & d > \bar{d}. \end{cases} \tag{36}$$

Note that $x \sim 1$ is the stationary limit, and $X(t,t') \sim 1$ in this case.

3.3. *Subcritical dynamics*

Again, the occurrence of a phase transition is required, and the calculations are performed for $d > d_c$. The asymptotic behaviour of ψ is given by

$$\psi(t) \sim \frac{f(t)}{M_{\text{eq}}^4}, \tag{37}$$

where

$$M_{\text{eq}}^2 := 1 - \frac{T}{T_c} \tag{38}$$

is the square of the static magnetization.

In the stationary regime, $1 \sim \tau \ll t'$, the autocorrelation,

$$C(t,t') \sim M_{\text{eq}}^2 + \left(1 - M_{\text{eq}}^2\right) C_{\text{eq},c}(\tau), \tag{39}$$

and the response function,

$$R(t,t') \sim f\left(\frac{\tau}{2}\right) \mathrm{e}^{-\hat{J}(k_c)\tau}, \tag{40}$$

depend on τ, and the fluctuation–dissipation theorem is asymptotically satisfied.

In the ageing time scale, $1 \ll \tau \sim t'$, the autocorrelation,

$$C(t,t') \sim M_{\text{eq}}^2 \left[\frac{4x}{(x+1)^2}\right]^{\frac{\gamma_p}{2}}, \tag{41}$$

and the response function,

$$R(t,t') \sim K_p 2^{\gamma_p} t'^{-\gamma_p} x^{\frac{\gamma_p}{2}} (x-1)^{-\gamma_p}, \tag{42}$$

are not invariant under time translation. One may calculate

$$\lim_{\tau\to\infty} \lim_{t'\to\infty} C(t,t') = M_{\text{eq}}^2 = 1 - \frac{T}{T_c}, \tag{43}$$

which is analogous to the Edwards–Anderson order parameter. This is a connection between the two time scales, and one can also interpolate the autocorrelation as

$$C(t,t') \sim \left(1 - M_{\text{eq}}^2\right) C_{\text{eq},c}(\tau) + M_{\text{eq}}^2 \left[\frac{4x}{(x+1)^2}\right]^{\frac{\gamma_p}{2}}. \tag{44}$$

The fluctuation–dissipation ratio is

$$X(t,t') \sim \frac{2TK_p}{\gamma_p M_{\text{eq}}^2} t'^{1-\gamma_p} \left(\frac{x+1}{x-1}\right)^{1+\gamma_p}. \tag{45}$$

4. Conclusions

In this work, the Langevin dynamics of a d-dimensional mean spherical model on a hypercubic lattice with nearest-neighbour (J and RJ) interactions and the addition of extra next-nearest-neighbour (SJ) interactions along $m \leqslant d$ directions was analysed. For $S < 0$ there is a scenario of competition between ferromagnetic and antiferromagnetic interactions, with the occurrence of an ordered modulated region in the phase diagram (in terms of the temperature T of the heat bath and the competition parameter $p = -S/R$). The asymptotic expressions (for large values of time t') for the autocorrelation, $C(t,t')$, and the response function, $R(t,t')$, with $t > t'$, were obtained, and the validity of the fluctuation–dissipation ratio, $X(t,t') = TR(t,t')/\partial_{t'}C(t,t')$, was checked.

The addition of competing interactions does not change the qualitative dynamical behaviour as compared to the ferromagnetic case (case 5 in this work), which has been analysed in detail by Godrèche and Luck [2]. The supercritical dynamics is trivial. The asymptotic forms of the two-time functions are translational invariant, $X(t,t') \sim 1$, and the system reaches equilibrium in a finite time. In the critical and subcritical cases, one is led to consider two distinct natural time scales: (i) for $1 \sim \tau \ll t'$, the two-time functions depend on the difference $\tau = t - t'$ only, and the fluctuation–dissipation theorem holds; (ii) if $1 \ll \tau \sim t'$, in general both the autocorrelation and the response function[1] depend on t and t' (instead of depending on τ only). This lack of translational invariance leads to violations of the fluctuation–dissipation theorem, and to a system that ages with time.

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Appendix A

The simple ferromagnetic model (case 5 in the classification of this work) is used in order to give a detailed account of the calculations. Results for the other cases can be obtained by analogous manipulations.

Lower critical dimension

The lower critical dimension, d_c, is established by the spherical constraint (5) at the critical value $\mu = \mu_c$,

$$\hat{J}(k_c) - \hat{J}(k) = J\left[c_2\sum_{i=1}^{m}(k_i - q_c)^2 + \sum_{i=m+1}^{d}k_i^2\right] - \frac{Jc_3}{3}\sum_{i=1}^{m}(k_i - q_c)^3 + \frac{J}{12}\left[c_4\sum_{i=1}^{m}(k_i - q_c)^4 + \sum_{i=m+1}^{d}k_i^4\right] + \cdots, \tag{A.1}$$

where $c_2 := R\cos q_c + 4S\cos(2q_c)$, $c_3 := R\sin q_c + 8S\sin(2q_c)$ and $c_4 := -R\cos q_c - 16S\cos(2q_c)$. It is easy to see that $c_2 \geqslant 0$, and $c_2 = 0$ if and only if $R + 4S = 0$ (corresponding

[1] Except the response function in critical dynamics for $d > \overline{d}$.

to $q_c = 0$) and $R - 4S = 0$ (corresponding to $q_c = \pi$). In these cases c_3 also vanishes and therefore the fourth-order term becomes relevant to characterize the critical behaviour.

For the fifth case, the (inverse) critical temperature is

$$\begin{aligned}\beta(\mu_c) &= \frac{1}{J}\int_{B_\delta}\frac{\mathrm{d}^d k}{(2\pi)^d}\frac{1}{\sum_{i=1}^d k_i^2 + \mathcal{O}(\delta^3)} + \int_{\hat{\Lambda}\backslash B_\delta}\frac{\mathrm{d}^d k}{(2\pi)^d}\frac{1}{\hat{J}(k_c) - \hat{J}(k)} \\ &= \frac{1}{J}\int_{B_\delta}\frac{\mathrm{d}^d k}{(2\pi)^d}\frac{1}{\sum_{i=1}^d k_i^2} + \mathcal{O}(\delta) \\ &= \frac{1}{J}\frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}\int_0^\delta \frac{\mathrm{d}k k^{d-3}}{(2\pi)^d} + \mathcal{O}(\delta),\end{aligned} \tag{A.2}$$

where B_δ is an open ball of radius δ centred at $(0, \ldots, 0)$, and in the last step the (hyper) spherical coordinates were invoked. The integral converges for $d > 2$, establishing $d_c = 2$. In this work, $h = \mathcal{O}(x)$ means that h is of order x or less than it; by $h = o(x)$, it means that h is of order less than x.

Initial conditions

From equation (12), the autocorrelation in Fourier space at $t = t' = 0$ is given by

$$C_k(0, 0) = \frac{1}{N}\sum_{x,x'\in\Lambda_N}\langle S_x(0)S_{x'}(0)\rangle\,\mathrm{e}^{ik(x-x')}, \tag{A.3}$$

where

$$\langle S_x(0)S_{x'}(0)\rangle = \begin{cases}\langle S_x(0)\rangle\langle S_{x'}(0)\rangle, & x \neq x' \\ \left\langle S_x^2(0)\right\rangle, & x = x'\end{cases} \tag{A.4}$$

for an 'infinite temperature' condition. In this highly disordered situation, from the spherical constraint $C(t, t) = 1$ at $t = 0$, one has

$$\begin{aligned}N &= \sum_{x\in\Lambda_N}\left\langle S_x^2(0)\right\rangle \\ &= N\left\langle S_x^2(0)\right\rangle,\end{aligned} \tag{A.5}$$

from which $\left\langle S_x^2(0)\right\rangle = 1$.

Therefore, $\langle S_x(0)S_{x'}(0)\rangle = \delta_{x,x'}$, which yields $C_k(0, 0) = 1$.

Asymptotic behaviour of f

For large t, and choosing $\delta \ll 1$ such that $t^{-1/2} \ll \delta \ll t^{-1/4}$, one shows that (case 5)

$$\begin{aligned}f(t) &= \left[\frac{1}{\pi}\int_0^\pi \mathrm{d}k\,\mathrm{e}^{4Jt\cos k}\right]^d \\ &= \left[\frac{1}{\pi}\int_0^\delta \mathrm{d}k\,\mathrm{e}^{4Jt(1-\frac{k^2}{2}+\mathcal{O}(\delta^4))} + \frac{1}{\pi}\int_\delta^\pi \mathrm{d}k\,\mathrm{e}^{4Jt\cos k}\right]^d \\ &= \left[\frac{\mathrm{e}^{4Jt}}{\pi}\int_0^\delta \mathrm{d}k\,\mathrm{e}^{-2Jtk^2}(1 + \mathcal{O}(t\delta^4)) + \frac{1}{\pi}\int_\delta^\pi \mathrm{d}k\,\mathrm{e}^{4Jt\cos k}\right]^d \\ &= \left[\frac{\mathrm{e}^{4Jt}}{\pi}\frac{1}{\sqrt{2Jt}}\frac{\sqrt{\pi}}{2}(\mathrm{erf}(\sqrt{2Jt}\delta) + \mathcal{O}(t^{3/2}\delta^4)) + \mathcal{O}(\mathrm{e}^{4Jt\cos\delta})\right]^d \\ &\sim \frac{\mathrm{e}^{4Jdt}}{(8\pi Jt)^{\frac{d}{2}}}.\end{aligned} \tag{A.6}$$

In general, one has equation (23) with K_p and γ_p given in table 2.

Asymptotic behaviour of $\mathcal{L}[f](s)$

One should consider

$$\mathcal{L}[f](s) = \int_{[-\pi,\pi]^d} \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{\epsilon + 2\hat{J}(k_c) - 2\hat{J}(k)}, \tag{A.7}$$

where $\epsilon := s - 2\hat{J}(k_c) > 0$, and recall that $k_c = 0$ and $2\hat{J}(k) = 4J\sum_{i=1}^d \cos k_i$ in case 5. In order to obtain the Laplace transform of f in the vicinity of $2\hat{J}(k_c)$ (or $\epsilon \sim 0$), and using the same notation as in equation (A.2), this expression is rewritten in the form

$$\mathcal{L}[f](s) = \int_{B_\delta} \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{\epsilon + 2\hat{J}(k_c) - \left[2\hat{J}(k_c) - 2J\sum_{i=1}^d k_i^2 + \mathcal{O}(\delta^4)\right]} + \int_{[-\pi,\pi]^d\setminus B_\delta} \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{\epsilon + 2\hat{J}(k_c) - 2\hat{J}(k)}. \tag{A.8}$$

Changing the first term of (A.8) to (hyper) spherical coordinates, and since the second term is analytic in ϵ, it is possible to write

$$\mathcal{L}[f](s) = \frac{\epsilon^{\frac{d-2}{2}}}{(8\pi J)^{\frac{d}{2}}\,\Gamma\left(\frac{d}{2}\right)} \int_0^{\frac{2J\delta^2}{\epsilon}} \frac{\mathrm{d}k\, k^{\frac{d}{2}-1}}{k+1} + \mathcal{O}(\delta^2) + \sum_{j=1}^{\infty}(-1)^{j-1} A'_j \epsilon^{j-1}, \tag{A.9}$$

where

$$A'_j := \int_{[-\pi,\pi]^d\setminus B_\delta} \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{[2\hat{J}(k_c) - 2\hat{J}(k)]^j}. \tag{A.10}$$

Also, define

$$A_j := \int_{[-\pi,\pi]^d} \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{[2\hat{J}(k_c) - 2\hat{J}(k)]^j}. \tag{A.11}$$

From equation (5), it is easily seen that $A_1 = \frac{1}{2T_c}$.

Let $d = 2q$. For even dimensions, and choosing $\epsilon \ll \delta^2 \ll 1$ such that $\epsilon \ln\delta \ll \delta^2$ and $\delta^2 \ll |\epsilon \ln \epsilon|$ (a possible choice is given by $\delta^2 = \epsilon\sqrt{|\ln\epsilon \ln\delta|}$), the integral in equation (A.9) is

$$\epsilon^{q-1}\int_0^{\frac{2J\delta^2}{\epsilon}} \frac{\mathrm{d}k\, k^{q-1}}{k+1} = \epsilon^{q-1}\int_1^{1+\frac{2J\delta^2}{\epsilon}} \frac{\mathrm{d}u}{u}\sum_{m=0}^{q-1}\binom{q-1}{m}(-1)^{q-1-m}u^m = \begin{cases} -\ln\epsilon + \mathcal{O}(\ln\delta^2), & q=1 \quad (d=2) \\ \epsilon\ln\epsilon + \mathcal{O}(\delta^2), & q=2 \quad (d=4) \\ (-1)^q\epsilon^{q-1}\ln\epsilon + \mathcal{O}(\delta^{2(q-1)}), & q>2 \quad (d>4). \end{cases} \tag{A.12}$$

On the other hand, for non-even dimensions $d \in (0,2)$, one sees that

$$\int_0^{\frac{2J\delta^2}{\epsilon}} \frac{\mathrm{d}k\, k^{\frac{d}{2}-1}}{k+1} = \Gamma\left(\frac{d}{2}\right)\Gamma\left(1-\frac{d}{2}\right) - \int_{\frac{2J\delta^2}{\epsilon}}^{\infty} \frac{\mathrm{d}k\, k^{\frac{d}{2}-1}}{k+1} = \Gamma\left(\frac{d}{2}\right)\Gamma\left(1-\frac{d}{2}\right) + \mathcal{O}\left(\left[\frac{\epsilon}{\delta^2}\right]^{|1-d/2|}\right). \tag{A.13}$$

An analytic continuation from $(0,2)$ to $\mathbb{R}\setminus\mathbb{Z}$ using the functional equation (A.13) leads to

$$\int_0^{\frac{2J\delta^2}{\epsilon}} \frac{\mathrm{d}k\, k^{\frac{d}{2}-1}}{k+1} \sim \Gamma\left(\frac{d}{2}\right)\Gamma\left(1-\frac{d}{2}\right), \qquad d \in \mathbb{R}\setminus\mathbb{Z}. \tag{A.14}$$

Therefore,

$$\mathcal{L}[f](s) = \sum_{j=1}^{\infty}(-1)^{j-1}A'_j\epsilon^{j-1} + \mathcal{O}(\delta^2)$$
$$+\begin{cases}(8\pi J)^{-\frac{d}{2}}\Gamma\left(1-\frac{d}{2}\right)\epsilon^{-\frac{2-d}{2}} + \mathcal{O}(\delta^{-(2-d)}), & d<2\\ (8\pi J)^{-1}(-\ln\epsilon) + \mathcal{O}(\ln\delta^2), & d=2\\ (8\pi J)^{-\frac{d}{2}}\Gamma\left(1-\frac{d}{2}\right)\epsilon^{\frac{d-2}{2}} + \mathcal{O}(\delta^{d-2}), & 2<d<4\\ (8\pi J)^{-2}\epsilon\ln\epsilon + \mathcal{O}(\delta^2), & d=4\\ \mathcal{O}(\delta^2), & d>4.\end{cases} \tag{A.15}$$

One should now analyse the behaviour of $\sum_{j=1}^{\infty}(-1)^{j-1}A'_j\epsilon^{j-1}$ for $\epsilon \sim 0$. First, note that A_j (equation (A.11)) is finite for $d > 2j$. In this case

$$\begin{aligned}A'_j &= A_j - \int_{B_\delta}\frac{d^dk}{(2\pi)^d}\frac{1}{[2\hat{J}(k_c) - 2\hat{J}(k)]^j}\\ &= A_j - \frac{2\pi^{\frac{d}{2}}}{(2\pi)^d\Gamma\left(\frac{d}{2}\right)(2J)^j}\int_0^\delta dk\, k^{d-1-2j} + \mathcal{O}(\delta^2)\\ &= A_j + \mathcal{O}(\delta^{d-2j}) + \mathcal{O}(\delta^2).\end{aligned} \tag{A.16}$$

For $d \leqslant 2j$ the integral A_j diverges, and one should evaluate the asymptotic behaviour of $\epsilon^{j-1}A'_j$ to add to (A.15) and, therefore, characterize $\mathcal{L}[f](s)$ for $s \sim 2\hat{J}(k_c)$ (or $\epsilon \sim 0$). Using $\cos x \leqslant 1 - \frac{x^2}{\pi^2}$ for $x \in [0, \pi] \subset \mathbb{R}$,

$$\begin{aligned}|\epsilon^{j-1}A'_j| &\leqslant \left|\epsilon^{j-1}\int_{B_\delta}\frac{d^dk}{(2\pi)^d}\frac{1}{\left[4Jd - 4J\left(d - \sum_{i=1}^{d}\frac{k_i^2}{\pi^2}\right)\right]^j} + \mathcal{O}(\delta^2)\right|\\ &\leqslant \frac{\epsilon^{j-1}}{(4\pi^{-2}J)^j(2\pi)^d}\frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}\left|\int_\delta^{\pi\sqrt{d}}dk\, k^{d-1-2j}\right| + \mathcal{O}(\delta^2)\\ &\leqslant \begin{cases}\frac{2\pi^j}{(4\pi^{-2}J)^j(2\pi)^{2j}\Gamma(j)}\epsilon^{j-1}|\ln\delta| + \mathcal{O}(\epsilon^{j-1}) + \mathcal{O}(\delta^2), & d=2j\\ \frac{2\pi^{\frac{d}{2}}(2\pi)^{-d}\delta^{d-2}}{(4\pi^{-2}J)^j\Gamma\left(\frac{d}{2}\right)(2j-d)}\left(\frac{\epsilon}{\delta^2}\right)^{j-1} + \mathcal{O}(\epsilon^{j-1}) + \mathcal{O}(\delta^2), & d<2j.\end{cases}\end{aligned} \tag{A.17}$$

Therefore,

$$\sum_{j=1}^{\infty}A'_j\epsilon^{j-1} = \begin{cases}\mathcal{O}(\delta^{-(2-d)}), & d<2\\ \mathcal{O}(-\ln\delta^2), & d=2\\ A_1 + \mathcal{O}(\delta^{d-2}), & 2<d<4\\ A_1 + \mathcal{O}(\delta^2), & d=4\\ A_1 - A_2\epsilon + \mathcal{O}(\delta^{d-4}), & d>4.\end{cases} \tag{A.18}$$

Combining equations (A.15) and (A.18),

$$\mathcal{L}[f](s) \sim \begin{cases}(8\pi J)^{-\frac{d}{2}}\Gamma\left(1-\frac{d}{2}\right)\epsilon^{-\frac{2-d}{2}}, & d<2\\ (8\pi J)^{-1}(-\ln\epsilon), & d=2\\ A_1 - (8\pi J)^{-\frac{d}{2}}\left|\Gamma\left(1-\frac{d}{2}\right)\right|\epsilon^{\frac{d-2}{2}}, & 2<d<4\\ A_1 - (8\pi J)^{-2}(-\epsilon\ln\epsilon), & d=4\\ A_1 - A_2\epsilon, & d>4.\end{cases} \tag{A.19}$$

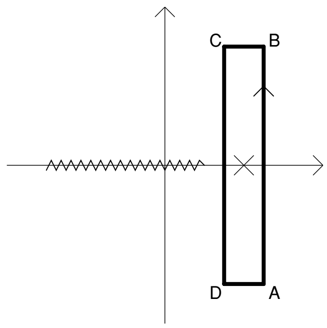

Figure A1. Contour for the integration—supercritical case.

Asymptotic behaviour of ψ—general comments

In this section the asymptotic behaviour of ψ,

$$\psi(t) = \frac{1}{2\pi \mathrm{i}} \int_{\sigma-\mathrm{i}\infty}^{\sigma+\mathrm{i}\infty} \mathrm{d}s\, \mathrm{e}^{st} \mathcal{L}[\psi](s), \qquad \mathcal{L}[\psi](s) = \frac{\mathcal{L}[f](s)}{1 - 2T\mathcal{L}[f](s)}, \tag{A.20}$$

where σ is larger than the real part of any pole of the integrand, will be evaluated. First, one should note that the Laplace transform of f,

$$\mathcal{L}[f](s) = \int_{[-\pi,\pi]^d} \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{s - 2\hat{J}(k)}, \tag{A.21}$$

has a cut on $[\inf_{k\in[-\pi,\pi]^d}\{2\hat{J}(k)\}, 2\hat{J}(k_c)]$ in the complex s-plane. Furthermore, $\mathcal{L}[f]$ is a monotonically decreasing function of s, ranging from 0 to $\beta_c/2$ ($\beta_c/2$ being infinite in the absence of phase transition).

Asymptotic behaviour of the ψ-supercritical case

Let $P := \{y \in \mathbb{C} : y \text{ is pole of } \mathrm{e}^{st}\mathcal{L}[\psi](s)\}$ and let $\overline{p} := \sup_{y\in P}\{\mathrm{Re}\, y\}$. By the monotonicity of $\mathcal{L}[f](s)$, the denominator of $\mathcal{L}[\psi](s)$ (see (A.20)) runs the interval $\left[1 - \frac{T}{T_c}, 1\right]$ reaching each point only one time. Therefore, in the supercritical dynamics, $T > T_c$, equation (A.20) has a single pole, denoted henceforth by τ_p^{-1}.

By the residue theorem, choosing the contour indicated in figure A1, where $ABCD$ is a rectangle with vertices $c \pm \mathrm{i}R$ and $\sigma \pm \mathrm{i}R$ such that $2\hat{J}(k_c) < c < \overline{p} < \sigma$,

$$2\pi\mathrm{i}\, \mathrm{Res}\, \mathrm{e}^{st}\mathcal{L}[\psi](s) = \int_{\sigma-\mathrm{i}R}^{\sigma+\mathrm{i}R} \mathrm{d}s\, \mathrm{e}^{st}\mathcal{L}[\psi](s) + \mathrm{e}^{\mathrm{i}Rt}\int_{\sigma}^{c} \mathrm{d}y\, \mathrm{e}^{yt}\mathcal{L}[\psi](y+\mathrm{i}R)$$
$$+ \mathrm{i}\,\mathrm{e}^{ct}\int_{R}^{-R} \mathrm{d}y\, \mathrm{e}^{\mathrm{i}yt}\mathcal{L}[\psi](c+\mathrm{i}y) + \mathrm{e}^{-\mathrm{i}Rt}\int_{c}^{\sigma} \mathrm{d}y\, \mathrm{e}^{yt}\mathcal{L}[\psi](y-\mathrm{i}R). \tag{A.22}$$

It is easy to see that $\lim_{R\to\infty} |\mathcal{L}[\psi](y \pm \mathrm{i}R)| = 0$. Moreover, the third term is $\mathcal{O}(\mathrm{e}^{ct})$, which is negligible as compared with the first one (equal to $2\pi\mathrm{i}\psi(t)$ in the limit $R \to \infty$), that is $\mathcal{O}(\mathrm{e}^{\sigma t})$. Therefore,

$$\psi(t) \sim \mathrm{Res}\, \mathrm{e}^{st}\mathcal{L}[\psi](s) = -\frac{1}{4T^2}\frac{1}{\partial_s\mathcal{L}[\psi]\left(\tau_p^{-1}\right)}\mathrm{e}^{\frac{t}{\tau_p}}. \tag{A.23}$$

As the temperature gets close to T_c (from above), τ_p^{-1} becomes closer to $2\hat{J}(k_c)$ (hitting this point at T_c), which is one of the edges of the cut in the complex s-plane. Hence, in the vicinity of T_c, one has

$$\mathrm{Res}\, \mathrm{e}^{st}\mathcal{L}[\psi](s) \sim \lim_{s\sim 2\hat{J}(k_c)^+} [s - 2\hat{J}(k_c)]\, \mathrm{e}^{st}\mathcal{L}[\psi](s), \qquad T \sim T_c^+, \tag{A.24}$$

and $\mathcal{L}[\psi]$ can be replaced by its asymptotic formula

$$\mathcal{L}[\psi](s) \sim \begin{cases} \frac{F_p g_p}{\epsilon^{-\alpha_p} - 2T F_p g_p}, & d < d_c \\ \frac{F_p(-\ln\epsilon)}{1-2T F_p(-\ln\epsilon)}, & d = d_c \\ \frac{A_1 - F_p g_p \epsilon^{\alpha_p}}{1-2T A_1 + 2T F_p |g_p| \epsilon^{\alpha_p}}, & d_c < d < \overline{d} \\ \frac{A_1 - F_p(-\epsilon\ln\epsilon)}{1-2T A_1 + 2T F_p(-\epsilon\ln\epsilon)}, & d = \overline{d} \\ \frac{A_1 - A_2\epsilon}{1-2T A_1 + 2T A_2 \epsilon}, & d > \overline{d}, \end{cases} \tag{A.25}$$

which can be calculated from (24) and (A.20). If these results are inserted in (A.24), one finds

$$\tau_p^{-1} \sim \begin{cases} 2\hat{J}(k_c) + (2T F_p g_p)^{-\frac{1}{\alpha_p}}, & d < d_c \\ 2\hat{J}(k_c) + \exp\left(-\frac{1}{2T F_p}\right), & d = d_c \\ 2\hat{J}(k_c) + \left[-\frac{1}{2T F_p |g_p|}\left(1 - \frac{T}{T_c}\right)\right]^{\frac{1}{\alpha_p}}, & d_c < d < \overline{d} \\ 2\hat{J}(k_c) + \epsilon^*, & d = \overline{d} \\ 2\hat{J}(k_c) + \left(-\frac{1}{2T A_2}\right)\left(1 - \frac{T}{T_c}\right), & d > \overline{d}, \end{cases} \tag{A.26}$$

where ϵ^* is the least root of

$$\epsilon \ln\epsilon = \frac{1}{2T F_p}\left(1 - \frac{T}{T_c}\right). \tag{A.27}$$

The characteristic relaxation time, τ_{eq}, is related to τ_p by

$$\tau_{\mathrm{eq}}^{-1} = \tau_p^{-1} - 2\hat{J}(k_c). \tag{A.28}$$

Asymptotic behaviour of the ψ-critical case

The simple pole of $\mathcal{L}[\psi](s)$, which is isolated in the supercritical case, touches $2\hat{J}(k_c)$ (one of the edges of the cut) at $T = T_c$. Taking the integration contour as in figure A2, it is easy to show that

$$\int_{\sigma-\mathrm{i}R}^{\sigma+\mathrm{i}R} \mathrm{d}s\, \mathrm{e}^{st}\mathcal{L}[\psi](s) = \int_{\mathrm{GFEDC}} \mathrm{d}s\, \mathrm{e}^{st}\mathcal{L}[\psi](s), \tag{A.29}$$

since the contribution of the paths BC and GA vanishes in the limit $R \to \infty$.

For sufficiently large time the integral in (A.29) is dominated by the contribution due to the path FED. Therefore, substituting $\mathcal{L}[f](s)$ by its asymptotic form (24) is a suitable operation to evaluate the asymptotic form of ψ.

If $d_c < d < \overline{d}$, one has $\mathcal{L}[f](s) \sim A_1 - F_p|g_p|\epsilon^{\alpha_p}$. Hence, $\mathcal{L}[\psi](s) \sim \frac{A_1^2}{F_p|g_p|}\epsilon^{-\alpha_p}$ and

$$\psi(t) \sim \frac{A_1^2}{F_p|g_p|}\frac{1}{2\pi\mathrm{i}}\int_{\mathrm{GFEDC}} \mathrm{d}s\, \mathrm{e}^{st}\left[s - 2\hat{J}(k_c)\right]^{-\alpha_p}. \tag{A.30}$$

The integral in (A.30) is a Gamma function with Hankel's contour. Therefore,

$$\psi(t) \sim \frac{A_1^2}{F_p|g_p|}\frac{t^{\alpha_p - 1}\mathrm{e}^{2\hat{J}(k_c)t}}{\Gamma(\alpha)}. \tag{A.31}$$

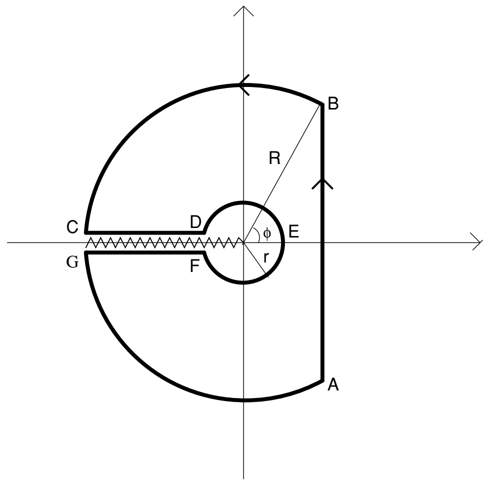

Figure A2. Contour for integration—critical case.

For $d = \overline{d}$, $\mathcal{L}[f](s) \sim A_1 - F_p(-\epsilon \ln \epsilon)$, and, therefore, $\mathcal{L}[\psi](s) \sim -\frac{A_1^2}{F_p \epsilon \ln \epsilon}$. The integration contour is again the one shown in figure A2. Since the contribution due to the arc FED is zero (letting its radius to zero), one finds

$$\psi(t) \sim -\frac{A_1}{2T_c F_p} \frac{\mathrm{e}^{2\hat{J}(k_c)t}}{2\pi \mathrm{i}} \left[\int_\infty^0 \frac{\mathrm{d}(r\,\mathrm{e}^{-\mathrm{i}\pi})\,\mathrm{e}^{tr\,\mathrm{e}^{-\mathrm{i}\pi}}}{r\,\mathrm{e}^{-\mathrm{i}\pi}\,(\ln r - \mathrm{i}\pi)} + \int_0^\infty \frac{\mathrm{d}(r\,\mathrm{e}^{\mathrm{i}\pi})\mathrm{e}^{tr\mathrm{e}^{\mathrm{i}\pi}}}{r\,\mathrm{e}^{\mathrm{i}\pi}\,(\ln r + \mathrm{i}\pi)} \right]$$
$$= \frac{A_1}{2T_c F_p} \mathrm{e}^{2\hat{J}(k_c)t} \int_0^\infty \frac{\mathrm{d}r\,\mathrm{e}^{-rt}}{r\left(\ln^2 r + \pi^2\right)}. \tag{A.32}$$

Adopting the change of variables $r \to \mathrm{e}^{\pi r}$, and integrating by parts, one has

$$\psi(t) \sim \frac{A_1}{2T_c F_p} \frac{\mathrm{e}^{2\hat{J}(k_c)t}}{\pi} \int_{-\infty}^\infty \frac{\mathrm{d}r \exp\left(-t\,\mathrm{e}^{\pi r}\right)}{r^2 + 1}$$
$$= \frac{A_1}{2T_c F_p} \frac{\mathrm{e}^{2\hat{J}(k_c)t}}{\pi} \left\{ \exp(-t\,\mathrm{e}^{\pi r}) \tan^{-1} r|_{-\infty}^{\infty} + \pi t \int_{-\infty}^\infty \mathrm{d}r \tan^{-1} r \exp(\pi r - t\,\mathrm{e}^{\pi r}) \right\}$$
$$= \frac{A_1}{2T_c F_p} \frac{\mathrm{e}^{2\hat{J}(k_c)t}}{\pi} \left\{ \frac{\pi}{2} - \int_0^\infty \mathrm{d}u\,\mathrm{e}^{-u} \tan^{-1}\left[\frac{1}{\pi} \ln\left(\frac{t}{u}\right)\right] \right\}$$
$$= \frac{A_1}{2T_c F_p} \frac{\mathrm{e}^{2\hat{J}(k_c)t}}{\pi} \left\{ \frac{\pi}{2} - \int_0^\infty \mathrm{d}u\,\mathrm{e}^{-u} \left[\frac{\pi}{2} - \frac{\pi}{\ln\left(\frac{t}{u}\right)} + \mathcal{O}(\ln^{-3} t) \right] \right\}$$
$$= \frac{A_1}{2T_c F_p} \frac{\mathrm{e}^{2\hat{J}(k_c)t}}{\ln t} \left[\int_0^\infty \mathrm{d}u \frac{\mathrm{e}^{-u}}{1 - \frac{\ln u}{\ln t}} + \mathcal{O}(\ln^{-4} t) \right]. \tag{A.33}$$

The application $\mathrm{e}^{-u}\left(1 - \frac{\ln u}{\ln t}\right)^{-1}$, as a function of u, is defined on the interval $(0, \infty)$ almost everywhere, and the integral

$$A(t) := \int_0^\infty \mathrm{d}u \frac{\mathrm{e}^{-u}}{1 - \frac{\ln u}{\ln t}} \tag{A.34}$$

is understood as the Cauchy principal value:

$$A(t) := \text{v.p.} \int_0^\infty \mathrm{d}u \frac{\mathrm{e}^{-u}}{1-\frac{\ln u}{\ln t}}. \tag{A.35}$$

In the next section it will be demonstrated that $\lim_{t\to\infty} A(t) = 1$. Using this result, one finds

$$\psi(t) \sim \frac{A_1}{2T_c F_p} \frac{\mathrm{e}^{2\hat{J}(k_c)t}}{\ln t} \tag{A.36}$$

for $d = \overline{d}$.

If $d > \overline{d}$, $\mathcal{L}[f](s) \sim A_1 - A_2\epsilon$ and $\mathcal{L}[\psi](s) \sim \frac{A_1^2}{A_2(s-2\hat{J}(k_c))}$. Using again the contour of figure A2, this leads to

$$\psi(t) \sim \frac{A_1^2}{A_2} \frac{1}{2\pi \mathrm{i}} \int_{\mathrm{JEFGK}} \mathrm{d}s \frac{\mathrm{e}^{st}}{s-2\hat{J}(k_c)}, \tag{A.37}$$

with the contribution due to the paths GF and DC being negligible as compared to the one due to FED. Letting its radius to zero, one sees that

$$\psi(t) \sim \frac{A_1^2}{A_2} \mathrm{e}^{2\hat{J}(k_c)t}. \tag{A.38}$$

The results are summarized by

$$\psi(t) \sim \begin{cases} \frac{A_1^2}{F_p|g|\Gamma(\alpha_p)} \frac{\mathrm{e}^{2\hat{J}(k_c)t}}{t^{1-\alpha_p}} & d_c < d < \overline{d} \\ \frac{A_1}{2T_c F_p} \frac{\mathrm{e}^{2\hat{J}(k_c)t}}{\ln t} & d = \overline{d} \\ \frac{A_1^2}{A_2} \mathrm{e}^{2\hat{J}(k_c)t} & d > \overline{d}. \end{cases} \tag{A.39}$$

Proof of $\lim_{t\to\infty} A(t) = 1$

Considering $t \gg e$, the integral

$$A(t) = \text{v.p.} \int_0^\infty \mathrm{d}u \frac{\mathrm{e}^{-u}}{1-\frac{\ln u}{\ln t}} = A^{(1)}(t) + A^{(2)}(t) + A^{(3)}(t) \tag{A.40}$$

is divided into three parts such that

$$A^{(1)}(t) := \int_0^{\frac{1}{\ln t}} \mathrm{d}u \frac{\mathrm{e}^{-u}}{1-\frac{\ln u}{\ln t}} = \ln t \int_0^{\frac{1}{\ln t}} \mathrm{d}u \frac{\mathrm{e}^{-u}}{\ln t + |\ln u|}. \tag{A.41}$$

Therefore,

$$\begin{aligned} |A^{(1)}(t)| &\leqslant \frac{\ln t}{\ln t + \ln\ln t} \int_0^{\frac{1}{\ln t}} \mathrm{d}u\, \mathrm{e}^{-u} = \frac{\ln t}{\ln t + \ln\ln t}\left(1 - \mathrm{e}^{-\frac{1}{\ln t}}\right) \\ &= \frac{\ln t}{\ln t + \ln\ln t}\left[\frac{1}{\ln t} + \mathcal{O}(\ln^{-2} t)\right], \end{aligned} \tag{A.42}$$

and $\lim_{t\to\infty} A^{(1)}(t) = 0$.

The second term, $A^{(2)}(t)$, is responsible for the non-zero contribution of the asymptotic behaviour of $A(t)$,

$$A^{(2)}(t) := \int_{\frac{1}{\ln t}}^{\ln t} \mathrm{d}u \frac{\mathrm{e}^{-u}}{1-\frac{\ln u}{\ln t}} = \int_{\frac{1}{\ln t}}^{\ln t} \mathrm{d}u\, \mathrm{e}^{-u}\, [1 + o(1)] = \mathrm{e}^{-\frac{1}{\ln t}} - \mathrm{e}^{-\ln t} + o(1); \tag{A.43}$$

hence, $\lim_{t\to\infty} A^{(2)}(t) = 1$.

It remains to show that the third term, $A^{(3)}(t)$, is zero for $t \to \infty$. Separating into three parts,

$$A^{(3)}(t) := \text{v.p.} \int_{\ln t}^{\infty} \mathrm{d}u \frac{\mathrm{e}^{-u}}{1 - \frac{\ln u}{\ln t}} = A^{(3a)}(t) + A^{(3b)}(t) + A^{(3c)}(t), \tag{A.44}$$

it will be shown that each of them vanishes in the limit $t \to \infty$.

The first term,

$$A^{(3a)}(t) := \int_{\ln t}^{t\left(1-\frac{1}{\ln t}\right)} \mathrm{d}u \frac{\mathrm{e}^{-u}}{1 - \frac{\ln u}{\ln t}} = -t \ln t \int_{\frac{\ln t}{t}}^{1-\frac{1}{\ln t}} \mathrm{d}u \frac{\mathrm{e}^{-tu}}{\ln u}, \tag{A.45}$$

admits the bound

$$\begin{aligned} |A^{(3a)}(t)| &\leqslant t \ln t \int_{\frac{\ln t}{t}}^{1-\frac{1}{\ln t}} \mathrm{d}u \frac{\mathrm{e}^{-tu}}{|\ln u|} \leqslant \frac{t \ln t}{\left|\ln\left(1 - \frac{1}{\ln t}\right)\right|} \int_{\frac{\ln t}{t}}^{1-\frac{1}{\ln t}} \mathrm{d}u\, \mathrm{e}^{-tu} \\ &= \frac{t \ln t}{\left|\frac{1}{\ln t} + \mathcal{O}\left(\frac{1}{\ln^2 t}\right)\right|} \frac{1}{t} \left[\frac{1}{t} - \mathrm{e}^{-t\left(1-\frac{1}{\ln t}\right)}\right], \end{aligned} \tag{A.46}$$

from which $\lim_{t\to\infty} A^{(3a)}(t) = 0$.

The second term,

$$A^{(3b)}(t) := \text{v.p.} \int_{t\left(1-\frac{1}{\ln t}\right)}^{t\left(1+\frac{1}{\ln t}\right)} \mathrm{d}u \frac{\mathrm{e}^{-u}}{1 - \frac{\ln u}{\ln t}} = -t \ln t \, \text{v.p.} \int_{1-\frac{1}{\ln t}}^{1+\frac{1}{\ln t}} \mathrm{d}u \frac{\mathrm{e}^{-tu}}{\ln u}, \tag{A.47}$$

can be bounded by

$$\begin{aligned} |A^{(3b)}(t)| &\leqslant t \ln t \, \text{v.p.} \int_{1-\frac{1}{\ln t}}^{1+\frac{1}{\ln t}} \mathrm{d}u \frac{\mathrm{e}^{-tu}}{|\ln u|} \\ &\leqslant t\, \mathrm{e}^{-t\left(1-\frac{1}{\ln t}\right)} \ln t \text{ v.p.} \int_{1-\frac{1}{\ln t}}^{1+\frac{1}{\ln t}} \mathrm{d}u \frac{1}{|\ln u|} \\ &= t\, \mathrm{e}^{-t\left(1-\frac{1}{\ln t}\right)} \ln t \text{ v.p.} \int_{-\frac{1}{\ln t}}^{\frac{1}{\ln t}} \frac{\mathrm{d}u}{|\ln(1+u)|} \\ &= t\, \mathrm{e}^{-t\left(1-\frac{1}{\ln t}\right)} \ln t \text{ v.p.} \int_{-\frac{1}{\ln t}}^{\frac{1}{\ln t}} \frac{\mathrm{d}u}{|u|} \left[1 + \frac{|u|}{2} + \mathcal{O}(u^2)\right] \\ &= t\, \mathrm{e}^{-t\left(1-\frac{1}{\ln t}\right)} \ln t \left[\frac{1}{\ln t} + o(\ln^{-1} t)\right], \end{aligned} \tag{A.48}$$

and $\lim_{t\to\infty} A^{(3b)}(t) = 0$.

Finally, the third term,

$$A^{(3c)}(t) := \int_{t\left(1+\frac{1}{\ln t}\right)}^{\infty} \mathrm{d}u \frac{\mathrm{e}^{-u}}{1 - \frac{\ln u}{\ln t}} = -t \ln t \int_{1+\frac{1}{\ln t}}^{\infty} \mathrm{d}u \frac{\mathrm{e}^{-tu}}{\ln u}, \tag{A.49}$$

also goes to zero in the limit $t \to \infty$,

$$|A^{(3c)}(t)| \leqslant \frac{t \ln t}{\ln\left(1 + \frac{1}{\ln t}\right)} \int_{1+\frac{1}{\ln t}}^{\infty} \mathrm{d}u\, \mathrm{e}^{-tu} = \frac{t \ln t}{\frac{1}{\ln t} + \mathcal{O}(\ln^{-2} t)} \frac{\mathrm{e}^{-t\left(1+\frac{1}{\ln t}\right)}}{t}, \tag{A.50}$$

which implies $\lim_{t\to\infty} A^{(3c)}(t) = 0$, and completes the proof.

Asymptotic behaviour of the ψ-subcritical case

In the subcritical case, where $\mathcal{L}[f](s) \sim A_1 - h\left(\left[s - 2\hat{J}(k_c)\right]\right)$ for $s \sim 2\hat{J}(k_c)$, h being a continuous function for $s \geqslant 2\hat{J}(k_c)$, and $h(0) = 0$. The form of h is given in (24). It is easily seen that

$$\mathcal{L}[\psi](s) = \frac{1}{M_{\text{eq}}^4}\mathcal{L}[f](s) - \frac{TA_1}{M_{\text{eq}}^4 T_c} + \mathcal{O}([h(s - 2\hat{J}(k_c)]^2), \quad s \sim 2\hat{J}(k_c), \tag{A.51}$$

where

$$M_{\text{eq}}^2 := 1 - \frac{T}{T_c}. \tag{A.52}$$

In analogy to the manipulations in the critical case, and using the contour given in figure A2, one finds, for large times, that

$$\psi(t) = \frac{1}{M_{\text{eq}}^4}\int_{\text{GFEDC}} \mathrm{d}s\, \mathrm{e}^{st}\mathcal{L}[f](s) + \mathcal{O}([h(s - 2\hat{J}(k_c)]^2). \tag{A.53}$$

The integral in equation (A.53) can be written as

$$\psi(t) \sim \frac{1}{M_{\text{eq}}^4}\mathcal{L}^{-1}\mathcal{L}[f](t) = \frac{f(t)}{M_{\text{eq}}^4}. \tag{A.54}$$

Three auxiliary equations

The following three equations are useful in the calculations of the asymptotic behaviour of the two-time functions (autocorrelation and response function):

$$\text{(i)} \quad \frac{1}{2T_c} = \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-2\hat{J}(k_c)t} f(t). \tag{A.55}$$

This equation is the definition of critical temperature, and it is easily obtained by the definition of f.

$$\text{(ii)} \quad \frac{1}{2T} = \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-\frac{t}{\tau_p}} f(t) \quad (T > T_c). \tag{A.56}$$

From equation (A.20), and τ_p^{-1} being the simple pole of $\mathcal{L}[\psi](s)$, one has

$$\mathcal{L}[f](\tau_p^{-1}) = \frac{1}{2T}. \tag{A.57}$$

Therefore,

$$\int_0^\infty \mathrm{d}t\, \mathrm{e}^{-\frac{t}{\tau_p}} f(t) = \lim_{s\to\tau_p^{-1}} \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-st} f(t) = \lim_{s\to\tau_p^{-1}} \mathcal{L}[f](s) = \frac{1}{2T}, \tag{A.58}$$

which is the desired result.

$$\text{(iii)} \quad \frac{1}{2T_c M_{\text{eq}}^2} = \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-2\hat{J}(k_c)t} \psi(t) \quad (T < T_c). \tag{A.59}$$

For $T < T_c$, one sees that $\mathcal{L}[f](s) \sim A_1 + h(s - 2\hat{J}(k_c))$ for $s \sim 2\hat{J}(k_c)^+$, where h is a continuous application of $s - 2\hat{J}(k_c)$, and therefore, $h(0) = 0$. Hence, by this expansion and from equation (A.20),

$$\begin{aligned}\lim_{s\to 2\hat{J}(k_c)^+} \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-st}\psi(t) &= \lim_{s\to 2\hat{J}(k_c)^+} \mathcal{L}[\psi](s) \\ &= \lim_{s\to 2\hat{J}(k_c)^+} \frac{\mathcal{L}[f](s)}{1 - 2T\mathcal{L}[f](s)} = \frac{A_1}{1 - 2TA_1}.\end{aligned} \tag{A.60}$$

The desired result follows from $A_1 = \frac{1}{2T_c}$.

Autocorrelation, response function and fluctuation–dissipation ratio

The autocorrelation (21),

$$C(t,t') = \frac{1}{\sqrt{\psi(t)\psi(t')}}\left[f\left(\frac{t+t'}{2}\right) + 2T\int_0^{t'} \mathrm{d}y\, f\left(\frac{t+t'}{2} - y\right)\psi(y)\right]$$
$$= \frac{1}{\sqrt{\psi(t'+\tau)\psi(t')}}\left[f\left(t'+\frac{\tau}{2}\right) + 2T\int_0^{t'} \mathrm{d}y\, f\left(t'-y+\frac{\tau}{2}\right)\psi(y)\right], \tag{A.61}$$

the response function,

$$R(t,t') = f\left(\frac{\tau}{2}\right)\sqrt{\frac{\psi(t')}{\psi(t)}}, \tag{A.62}$$

and the fluctuation–dissipation ratio,

$$X(t,t') = \frac{TR(t,t')}{\partial_{t'}C(t,t')}, \tag{A.63}$$

display different behaviours, which depend on the temperature and the chosen time scale. The calculations of these functions will be divided into three parts, each of them corresponding to different choices of temperature. In each part, distinct time scales, leading to the dynamical behaviour of the two-time function, will be considered. The notation

$$x := \frac{t}{t'} \tag{A.64}$$

will be used.

Let ψ_a be the asymptotic form of ψ. In other words,

$$\psi(t) \sim \psi_a(t) = \begin{cases} D_> \mathrm{e}^{t/\tau_p}, & T > T_c \\ D_1 = \frac{\mathrm{e}^{2\hat{J}(k_c)t}}{t^{2-\gamma_p}}, & T = T_c \quad \text{and} \quad d_c < d < \overline{d} \\ D_2 = \frac{\mathrm{e}^{2\hat{J}(k_c)t}}{\ln t}, & T = T_c \quad \text{and} \quad d = \overline{d} \\ D_3 = \mathrm{e}^{2\hat{J}(k_c)t}, & T = T_c \quad \text{and} \quad d > \overline{d} \\ \frac{f(t)}{M_{\mathrm{eq}}^4}, & T < T_c. \end{cases} \tag{A.65}$$

Choosing an $\epsilon > 0$ such that $1 \ll \epsilon t' \ll t'$, one can write (A.61) as

$$C(t,t') \sim \frac{1}{\sqrt{\psi_a(t)\psi_a(t')}}\left[\frac{K_p\,\mathrm{e}^{\hat{J}(k_c)(2t'+\tau)}}{\left(t'+\frac{\tau}{2}\right)^{\gamma_p}} + 2T\int_0^{\epsilon t'} \mathrm{d}y \frac{K_p\,\mathrm{e}^{\hat{J}(k_c)(2t'+\tau-2y)}}{\left(t'+\frac{\tau}{2}-y\right)^{\gamma_p}}\psi(y)\right.$$
$$\left. + 2T\int_{\epsilon t'}^{t'} \mathrm{d}y f\left(t'+\frac{\tau}{2}-y\right)\psi_a(y)\right]$$
$$= \frac{1}{\sqrt{\psi_a(t)\psi_a(t')}}\left[\frac{K_p\,\mathrm{e}^{\hat{J}(k_c)(2t'+\tau)}}{\left(t'+\frac{\tau}{2}\right)^{\gamma_p}} + \frac{2TK_p\,\mathrm{e}^{\hat{J}(k_c)(2t'+\tau)}}{\left(t'+\frac{\tau}{2}\right)^{\gamma_p}}\right.$$
$$\left. \times \int_0^{\epsilon t'} \mathrm{d}y\,\mathrm{e}^{-2\hat{J}(k_c)y}\psi(y) + 2T\int_{\frac{\tau}{2}}^{(1-\epsilon)t'+\frac{\tau}{2}} \mathrm{d}y\, f(y)\psi_a\left(t'+\frac{\tau}{2}-y\right)\right]. \tag{A.66}$$

Since the function $w(y) = \psi(y)\,\mathrm{e}^{-2\hat{J}(k_c)y}$ is positive and non-increasing ($\mathrm{d}w(y)/\mathrm{d}y \leqslant 0$) on the real line, equation (A.66) can be written as

$$C(t,t') \sim \frac{1}{\sqrt{\psi_a(t)\psi_a(t')}}\left[\mathcal{O}\left(\frac{\epsilon t'\mathrm{e}^{\hat{J}(k_c)(2t'+\tau)}}{(t'+\tau/2)^{\gamma_p}}\right) + 2T\int_{\frac{\tau}{2}}^{(1-\epsilon)t'+\frac{\tau}{2}} \mathrm{d}y\, f(y)\psi_a\left(t'+\frac{\tau}{2}-y\right)\right]. \tag{A.67}$$

Supercritical dynamics

By equations (A.65) and (A.67), one finds

$$C(t,t') \sim \mathcal{O}\left(\frac{\epsilon t' \mathrm{e}^{(\hat{J}(k_c)-\frac{1}{2\tau_p})(2t'+\tau)}}{\left(t'+\frac{\tau}{2}\right)^{\gamma_p}}\right) + 2T\int_{\frac{\tau}{2}}^{(1-\epsilon)t'+\tau/2} \mathrm{d}y\, f(y)\,\mathrm{e}^{-y/\tau_p}$$
$$= \mathcal{O}\left(\frac{\epsilon t' \mathrm{e}^{(\hat{J}(k_c)-\frac{1}{2\tau_p})(2t'+\tau)}}{\left(t'+\frac{\tau}{2}\right)^{\gamma_p}}\right) + 2T\int_{\frac{\tau}{2}}^{\infty} \mathrm{d}y\, f(y)\,\mathrm{e}^{-y/\tau_p}$$
$$- 2T\int_{(1-\epsilon)t'+\tau/2}^{\infty} \mathrm{d}y\, f(y)\,\mathrm{e}^{-y/\tau_p}. \qquad \text{(A.68)}$$

In the asymptotic limit $t' \sim \infty$, the first and third terms are negligible as compared with the second one, which is $\mathcal{O}(1)$ (see (A.56)). Therefore, one has

$$C(t,t') \sim T\int_{\tau}^{\infty} \mathrm{d}y\, f\left(\frac{y}{2}\right)\mathrm{e}^{-\frac{y}{2\tau_p}}. \qquad \text{(A.69)}$$

By equations (A.23) and (A.62), the response function is

$$R(t,t') \sim f\left(\frac{\tau}{2}\right)\mathrm{e}^{-\frac{\tau}{2\tau_p}}. \qquad \text{(A.70)}$$

Using equations (A.69) and (A.70), one checks the fluctuation–dissipation theorem,

$$X(t,t') \sim Tf\left(\frac{\tau}{2}\right)\mathrm{e}^{-\frac{\tau}{2\tau_p}}\left[-\frac{\partial}{\partial\tau}T\int_{\tau}^{\infty}\mathrm{d}y\, f\left(\frac{y}{2}\right)\mathrm{e}^{-\frac{y}{2\tau_p}}\right]^{-1} = 1. \qquad \text{(A.71)}$$

Critical dynamics

In the stationary regime ($1 \sim \tau \ll t'$), one may rewrite (A.67) as

$$C(t,t') \sim \frac{1}{\sqrt{\psi_a(t)\psi_a(t')}}\left[\mathcal{O}\left(\frac{\epsilon t' \mathrm{e}^{\hat{J}(k_c)(2t'+\tau)}}{\left(t'+\frac{\tau}{2}\right)^{\gamma_p}}\right) + 2T_c\int_{\frac{\tau}{2}}^{\epsilon t'+\tau/2}\mathrm{d}y\, f(y)\right.$$
$$\left.\times\,\psi_a\left(t'+\frac{\tau}{2}-y\right) + 2T_c\int_{\epsilon t'+\tau/2}^{(1-\epsilon)t'+\tau/2}\mathrm{d}y\, f(y)\psi_a\left(t'+\frac{\tau}{2}-y\right)\right], \qquad \text{(A.72)}$$

which may be convenient for calculating the analytic asymptotic form of the autocorrelation in this regime.

Critical dynamics ($d_c < d < \overline{d}$). By (A.65) and (A.72), in the stationary regime, one has

$$C(t,t') \sim [(t'+\tau)t']^{\frac{2-\gamma_p}{2}}\left[\mathcal{O}\left(\frac{\epsilon t'}{(t'+\tau/2)^{\gamma_p}}\right) + \frac{2T_c}{\left(t'+\frac{\tau}{2}\right)^{2-\gamma_p}}\int_{\frac{\tau}{2}}^{\epsilon t'+\tau/2}\mathrm{d}y\right.$$
$$\left.\times f(y)\,\mathrm{e}^{-2\hat{J}(k_c)y} + 2T_cK_p\int_{\epsilon t'+\tau/2}^{(1-\epsilon)t'+\tau/2}\frac{\mathrm{d}y}{y^{\gamma_p}\left(t'+\frac{\tau}{2}-y\right)^{2-\gamma_p}}\right].$$

Performing the change of variable $y \to 1/y$ in the last term, one finds

$$C(t,t') \sim [(t'+\tau)t']^{\frac{2-\gamma_p}{2}}\left[\mathcal{O}\left(\frac{\epsilon t'}{\left(t'+\frac{\tau}{2}\right)^{\gamma_p}}\right) + \frac{1}{\left(t'+\frac{\tau}{2}\right)^{2-\gamma_p}}C_{\mathrm{eq},c}(\tau)\right], \qquad \text{(A.73)}$$

where $C_{\mathrm{eq},c}$ is defined as

$$C_{\mathrm{eq},c}(\tau) = T_c\int_{\tau}^{\infty}\mathrm{d}y\, f\left(\frac{y}{2}\right)\mathrm{e}^{-\hat{J}(k_c)y}. \qquad \text{(A.74)}$$

Restricting the range of $\epsilon > 0$ to be such that $1 \ll \epsilon t' \ll (t')^{\min\{1, 2\gamma_p - 2\}}$ leads to

$$C(t, t') \sim C_{\mathrm{eq},c}(\tau). \tag{A.75}$$

Since the response function is

$$\begin{aligned} R(t, t') &\sim f\left(\frac{\tau}{2}\right)\sqrt{\frac{\mathrm{e}^{2\hat{J}(k_c)t'}/t'^{1-\alpha_p}}{\mathrm{e}^{2\hat{J}(k_c)t}/t^{1-\alpha_p}}} \\ &= f\left(\frac{\tau}{2}\right)\frac{\mathrm{e}^{-\hat{J}(k_c)\tau}}{\left(1+\frac{\tau}{t'}\right)^{1-\alpha_p}} \\ &\sim f\left(\frac{\tau}{2}\right)\mathrm{e}^{-\hat{J}(k_c)\tau}, \end{aligned} \tag{A.76}$$

it is also a function of τ only, and the fluctuation–dissipation theorem, $X(t, t') \sim 1$, is asymptotically obeyed.

In the ageing regime, from (A.67), one has

$$\begin{aligned} C(t, t') &\sim \left[\left(t'+\tau\right)t'\right]^{\frac{2-\gamma_p}{2}}\left[\mathcal{O}\left(\frac{\epsilon t'}{\left(t'+\frac{\tau}{2}\right)^{\gamma_p}}\right) + 2T_c K_p \int_{\frac{\tau}{2}}^{(1-\epsilon)t'+\tau/2} \frac{\mathrm{d}y}{y^{\gamma_p}\left(t'+\frac{\tau}{2}-y\right)^{2-\gamma_p}}\right] \\ &\sim \left[\left(t'+\tau\right)t'\right]^{\frac{2-\gamma_p}{2}}\left[\mathcal{O}\left(\frac{\epsilon t'}{\left(t'+\frac{\tau}{2}\right)^{\gamma_p}}\right) + \frac{2T_c K_p}{(\gamma_p-1)\left(t'+\frac{\tau}{2}\right)}\left(\frac{2t'}{\tau}\right)^{\gamma_p-1}\right]. \end{aligned} \tag{A.77}$$

Restricting $\epsilon > 0$ to be such that $1 \ll \epsilon(t'+\tau/2) \ll (t'/\tau)^{\gamma_p-1}(t'+\tau/2)^{\gamma_p}$, one finds

$$C(t, t') \sim \frac{2K_p T_c 2^{\gamma_p}}{\gamma_p - 1} t'^{1-\gamma_p} \frac{x^{1-\frac{\gamma_p}{2}}(x-1)^{1-\gamma_p}}{x+1}. \tag{A.78}$$

From equations (23) and (A.62), one calculates the response function,

$$\begin{aligned} R(t, t') &\sim f\left(\frac{\tau}{2}\right)\sqrt{\frac{\mathrm{e}^{2\hat{J}(k_c)t'}/t'^{1-\alpha_p}}{\mathrm{e}^{2\hat{J}(k_c)t}/t^{1-\alpha_p}}} \\ &= f\left(\frac{\tau}{2}\right)\mathrm{e}^{-\hat{J}(k_c)\tau} x^{\frac{1-\alpha_p}{2}} \\ &\sim 2^{\gamma_p} K_p t'^{-\gamma_p}(x-1)^{-\gamma_p} x^{\frac{1-\alpha_p}{2}}, \end{aligned} \tag{A.79}$$

and the fluctuation–dissipation ratio,

$$X(t, t') \sim \frac{(\gamma_p - 1)(x+1)^2}{(\gamma_p x + \gamma_p - 2)(x+1) - 2(x-1)}, \tag{A.80}$$

which comes from equations (A.78) and (A.79).

Critical dynamics ($d = \bar{d}$). In the stationary regime, $1 \sim \tau \ll t'$, for $d = \bar{d}$, one should proceed analogously as was done in the stationary regime of the case $d_c < d < \bar{d}$. Therefore, by choosing $\epsilon > 0$ such that $1 \ll \epsilon t' \ll (t'+\tau/2)^{\gamma_p}/\ln(t'+\tau/2)$, one finds, by equations (A.65) and (A.72), that

$$\begin{aligned} C(t, t') \sim \sqrt{\ln(t'+\tau)\ln t'}&\left[\mathcal{O}\left(\frac{\epsilon t'}{\left(t'+\frac{\tau}{2}\right)^{\gamma_p}}\right) + \frac{2T_c}{\ln\left(t'+\frac{\tau}{2}\right)}\right. \\ &\left.\times \int_{\frac{\tau}{2}}^{\epsilon t'+\tau/2} \mathrm{d}y\, f(y)\,\mathrm{e}^{-2\hat{J}(k_c)y} + 2T_c K_p \int_{\epsilon t'+\tau/2}^{(1-\epsilon)t'+\tau/2} \frac{\mathrm{d}y}{y^{\gamma_p}\ln\left(t'+\frac{\tau}{2}-y\right)}\right] \end{aligned}$$

$$= \sqrt{\ln(t'+\tau)\ln t'}\Bigg[\mathcal{O}\left(\frac{\epsilon t'}{\left(t'+\frac{\tau}{2}\right)^{\gamma_p}}\right) + \frac{2T_c}{\ln\left(t'+\frac{\tau}{2}\right)} \times \left(\int_{\frac{\tau}{2}}^{\infty} \mathrm{d}y\, f(y)\,\mathrm{e}^{-2\hat{J}(k_c)y} - \int_{\epsilon t'+\tau/2}^{\infty} \frac{K_p \mathrm{d}y}{y^{\gamma_p}}\right) + \mathcal{O}\left(\frac{1}{\epsilon t'}\int_{\epsilon t'+\tau/2}^{(1-\epsilon)+\tau/2} \frac{\mathrm{d}y}{y^{\gamma_p}}\right)\Bigg] \sim C_{\mathrm{eq},c}(\tau). \tag{A.81}$$

Furthermore, by equation (A.62), it can be shown that

$$R(t,t') \sim f\left(\frac{\tau}{2}\right)\sqrt{\frac{\mathrm{e}^{2\hat{J}(k_c)t'}/\ln t'}{\mathrm{e}^{2\hat{J}(k_c)t}/\ln(t'+\tau)}} = f\left(\frac{\tau}{2}\right)\frac{\mathrm{e}^{-\hat{J}(k_c)\tau}}{\sqrt{1+\frac{\ln(1+\tau/t')}{\ln t'}}} \sim f\left(\frac{\tau}{2}\right)\mathrm{e}^{-\hat{J}(k_c)\tau}, \tag{A.82}$$

ensuring that $X(t,t') \sim 1$ in the stationary time scale.

In the ageing regime, $1 \ll \tau \sim t'$, from (A.65) and (A.67), one finds

$$C(t,t') \sim \sqrt{\ln t \ln t'}\left[\mathcal{O}\left(\frac{\epsilon t'}{\left(t'+\frac{\tau}{2}\right)^{\gamma_p}}\right) + 2T_c K_p \int_{\frac{\tau}{2}}^{(1-\epsilon)t'+\tau/2} \frac{\mathrm{d}y}{y^{\gamma_p}\ln\left(t'+\frac{\tau}{2}-y\right)}\right]$$
$$= \sqrt{\ln t \ln t'}\left[\mathcal{O}\left(\frac{\epsilon t'}{\left(t'+\frac{\tau}{2}\right)^{\gamma_p}}\right) + 2T_c K_p \int_{\epsilon t'}^{t'} \frac{\mathrm{d}y}{\left(t'+\frac{\tau}{2}-y\right)^{\gamma_p}\ln y}\right]$$
$$= \sqrt{\ln t \ln t'}\left[\mathcal{O}\left(\frac{\epsilon t'}{\left(t'+\frac{\tau}{2}\right)^{\gamma_p}}\right) + \frac{2T_c K_p t'}{\ln t'}\int_{\epsilon}^{1} \frac{\mathrm{d}u}{\left(t'+\frac{\tau}{2}-t'u\right)^{\gamma_p}\left(1+\frac{\ln u}{\ln t'}\right)}\right], \tag{A.83}$$

where the change of variable $y = t'u$ was performed in the last step. Since the condition $\epsilon t' \gg 1$ is equivalent to $1 \gg -\ln\epsilon/\ln t' = |\ln\epsilon/\ln t'|$, one finds

$$C(t.t') \sim \sqrt{\ln t \ln t'}\left[\mathcal{O}\left(\frac{\epsilon t'}{\left(t'+\frac{\tau}{2}\right)^{\gamma_p}}\right) + \frac{2T_c K_p t'}{\ln t'}\int_{\epsilon}^{1} \frac{\mathrm{d}u}{\left(t'+\frac{\tau}{2}-t'u\right)^{\gamma_p}}\right], \tag{A.84}$$

which leads to

$$C(t,t') \sim \frac{2^{\gamma_p}T_c K_p}{\gamma_p - 1}(t')^{1-\gamma_p}\left[(x-1)^{1-\gamma_p} - (x+1)^{1-\gamma_p}\right]\sqrt{1+\frac{\ln x}{\ln t'}}. \tag{A.85}$$

The calculation of the response function is simpler,

$$R(t,t') \sim f\left(\frac{\tau}{2}\right)\mathrm{e}^{-\hat{J}(k_c)\tau}\sqrt{1+\frac{\ln x}{\ln t'}} \sim K_p 2^{\gamma_p} t'^{-\gamma_p}(x-1)^{-\gamma_p}\sqrt{1+\frac{\ln x}{\ln t'}}. \tag{A.86}$$

Therefore, the fluctuation–dissipation ratio is

$$X(t,t') \sim \frac{2(\gamma_p - 1)\ln t'}{2(\gamma_p - 1)\left[1+\left(\frac{x-1}{x+1}\right)^{\gamma_p}\right]\ln t' - (x-1)\left[1-\left(\frac{x-1}{x+1}\right)^{\gamma_p - 1}\right]}. \tag{A.87}$$

Critical dynamics ($d > \overline{d}$). For $d > \overline{d}$, by equations (A.65) and (A.67), the autocorrelation is

$$C(t,t') \sim \mathcal{O}\left(\frac{\epsilon t'}{\left(t'+\frac{\tau}{2}\right)^{\gamma_p}}\right) + 2T_c \int_{\frac{\tau}{2}}^{(1-\epsilon)t'+\tau/2} \mathrm{d}y\, f(y)\,\mathrm{e}^{-2\hat{J}(k_c)y}$$
$$\sim C_{\mathrm{eq},c}(\tau), \tag{A.88}$$

since $C_{eq,c}(\tau)$ is $\mathcal{O}(1)$, which is much larger than other terms in the asymptotic limit.

The response function is obtained from equations (A.39) and (A.62):

$$R(t,t') \sim f\left(\frac{\tau}{2}\right)\mathrm{e}^{-\hat{J}(k_c)\tau}. \tag{A.89}$$

The asymptotic expansions of the two-time functions in the stationary regime (for $\tau \sim 1$) are given by equations (A.88) and (A.89). In this case, one also finds that $X(t,t') \sim 1$. On the other hand, in the ageing scenario, for $\tau \gg 1$, using (A.67), these functions have the following asymptotic behaviour:

$$C(t,t') \sim \mathcal{O}\left(\frac{\epsilon t'}{\left(t'+\frac{\tau}{2}\right)^{\gamma_p}}\right) + 2T_c K_p \int_{\frac{\tau}{2}}^{(1-\epsilon)t'+\tau/2} \frac{\mathrm{d}y}{y^{\gamma_p}}$$
$$\sim \frac{2^{\gamma_p} T_c K_p}{\gamma_p - 1}(t')^{1-\gamma_p}\left[(x-1)^{1-\gamma_p} - (x+1)^{1-\gamma_p}\right] \tag{A.90}$$

and

$$R(t,t') \sim 2^{\gamma_p} K_p \tau^{-\gamma_p} = 2^{\gamma_p} K_p t'^{-\gamma_p}(x-1)^{-\gamma_p}. \tag{A.91}$$

In this situation, the fluctuation–dissipation ratio is violated with

$$X(t,t') \sim \frac{1}{1+\left(\frac{x-1}{x+1}\right)^{\gamma_p}}. \tag{A.92}$$

Subcritical dynamics

As in the critical dynamics, the two characteristic time scales (stationary and aging) are also present.

In the stationary case, $1 \sim \tau \ll t'$, from equations (23), (A.65) and (A.66), one has

$$C(t,t') \sim M_{\mathrm{eq}}^4\left[\left(t'+\tau\right)t'\right]^{\frac{\gamma_p}{2}}\left[\frac{1}{\left(t'+\frac{\tau}{2}\right)^{\gamma_p}} + \frac{2T}{\left(t'+\frac{\tau}{2}\right)^{\gamma_p}}\frac{1}{2T_c M_{\mathrm{eq}}^2}\right.$$
$$\left. + \frac{2T}{M_{\mathrm{eq}}^4}\int_{\frac{\tau}{2}}^{(1-\epsilon)t'+\tau/2}\frac{\mathrm{d}y\, f(y)\,\mathrm{e}^{-2\hat{J}(k_c)y}}{\left(t'+\frac{\tau}{2}-y\right)^{\gamma_p}}\right]$$
$$\sim \frac{M_{\mathrm{eq}}^2\left[\left(t'+\tau\right)t'\right]^{\frac{\gamma_p}{2}}}{\left(t'+\frac{\tau}{2}\right)^{\gamma_p}}\left[M_{\mathrm{eq}}^2 + \frac{T}{T_c} + \frac{2T\left(t'+\frac{\tau}{2}\right)^{\gamma_p}}{M_{\mathrm{eq}}^2}\right.$$
$$\left.\times\left(\int_{\frac{\tau}{2}}^{\epsilon t'+\tau/2}\frac{\mathrm{d}y\, f(y)\,\mathrm{e}^{-2\hat{J}(k_c)y}}{\left(t'+\frac{\tau}{2}\right)^{\gamma_p}} + \int_{\epsilon t'+\tau/2}^{(1-\epsilon)t'+\tau/2}\frac{\mathrm{d}y\, f(y)\,\mathrm{e}^{-2\hat{J}(k_c)y}}{\left(t'+\frac{\tau}{2}-y\right)^{\gamma_p}}\right)\right]$$
$$\sim M_{\mathrm{eq}}^2\left[1 + \frac{2T}{M_{\mathrm{eq}}^2}\frac{C_{\mathrm{eq},c}(\tau)}{2T_c} + \mathcal{O}\left(\int_{\epsilon t'+\tau/2}^{(1-\epsilon)t'+\tau/2}\frac{\left(t'+\frac{\tau}{2}\right)^{\gamma_p}\mathrm{d}y}{y^{\gamma_p}\left(t'+\frac{\tau}{2}-y\right)^{\gamma_p}}\right)\right]$$
$$= M_{\mathrm{eq}}^2 + \left(1-M_{\mathrm{eq}}^2\right)C_{\mathrm{eq},c}(\tau) + \mathcal{O}\left(\left(\frac{t'+\frac{\tau}{2}}{\epsilon t'}\right)^{\gamma_p}\int_{\epsilon t'+\tau/2}^{(1-\epsilon)t'+\tau/2}\frac{\mathrm{d}y}{y^{\gamma_p}}\right), \tag{A.93}$$

where equation (A.59) was invoked. By choosing $\epsilon > 0$ such that $1 \ll (t')^{\frac{\gamma_p}{2\gamma_p-1}} \ll \epsilon t' \ll t'$, one has

$$C(t,t') \sim M_{\rm eq}^2 + \left(1 - M_{\rm eq}^2\right) C_{{\rm eq},c}(\tau). \tag{A.94}$$

Using equation (A.54), the response function, described by

$$R(t,t') \sim f\left(\frac{\tau}{2}\right)\sqrt{\frac{f(t')}{f(t)}} \tag{A.95}$$

in both time scales, behaves as

$$R(t,t') \sim f\left(\frac{\tau}{2}\right) {\rm e}^{-\hat{J}(k_c)\tau}\left(1+\frac{\tau}{t'}\right)^{\frac{\gamma_p}{2}} = f\left(\frac{\tau}{2}\right){\rm e}^{-\hat{J}(k_c)\tau} + \mathcal{O}\left(\frac{\tau}{t'}\right) \tag{A.96}$$

in the stationary case. It is not difficult to see that in this case the fluctuation–dissipation theorem is valid, with $X(t,t') \sim 1$.

In the ageing regime, $1 \ll \tau \sim t'$, from equations (23) and (A.54) in (A.61), the autocorrelation can be written as

$$\begin{aligned} C(t,t') \sim M_{\rm eq}^4\left[\left(t'+\tau\right)t'\right]^{\frac{\gamma_2}{2}} &\left[\frac{1}{\left(t'+\frac{\tau}{2}\right)^{\gamma_p}} + \frac{2T}{\left(t'+\frac{\tau}{2}\right)^{\gamma_p}}\frac{1}{2T_c M_{\rm eq}^2}\right. \\ &\left. + \frac{2TK_p}{M_{\rm eq}^4}\int_{\frac{\tau}{2}}^{(1-\epsilon)t+\tau/2}\frac{{\rm d}y}{y^{\gamma_p}\left(t'+\frac{\tau}{2}-y\right)^{\gamma_p}}\right] \\ = M_{\rm eq}^2\left[\left(t'+\tau\right)t'\right]^{\frac{\gamma_2}{2}} &\left[\frac{1}{\left(t'+\frac{\tau}{2}\right)^{\gamma_p}} + \mathcal{O}\left(\frac{1}{(\epsilon t')^{\gamma_p}}\int_{\frac{\tau}{2}}^{(1-\epsilon)t'+\tau/2}\frac{{\rm d}y}{y^{\gamma_p}}\right)\right]. \end{aligned} \tag{A.97}$$

Taking $\epsilon > 0$ such that $(t'+\tau/2)\tau^{(1-\gamma_p)/\gamma_p} \ll \epsilon t' \ll t'$, one has

$$C(t,t') \sim M_{\rm eq}^2\left[\frac{4x}{(x+1)^2}\right]^{\frac{\gamma_p}{2}}. \tag{A.98}$$

The calculation of the response function is simpler:

$$R(t,t') \sim \frac{K_p {\rm e}^{\hat{J}(k_c)\tau}}{\left(\frac{\tau}{2}\right)^{\gamma_p}}{\rm e}^{-\hat{J}(k_c)\tau}x^{\frac{\gamma_p}{2}} = K_p 2^{\gamma_p} t'^{-\gamma_p} x^{\frac{\gamma_p}{2}}(x-1)^{-\gamma_p}. \tag{A.99}$$

From equations (A.98) and (A.99), the fluctuation–dissipation ratio is given by the asymptotic expression

$$X(t,t') \sim \frac{2TK_p}{\gamma_p M_{\rm eq}^2} t'^{1-\gamma_p}\left(\frac{x+1}{x-1}\right)^{1+\gamma_p}. \tag{A.100}$$

Technical note

From equations (A.61) and (17), with $t \to t' + \tau/2$, one finds another form for the autocorrelation,

$$\begin{aligned} C(t,t') &= \frac{1}{\sqrt{\psi(t)\psi(t')}}\left[\psi\left(t'+\frac{\tau}{2}\right) - 2T\int_{t'}^{t'+\frac{\tau}{2}}{\rm d}y\, f\left(t'+\frac{\tau}{2}-y\right)\psi(y)\right] \\ &= \frac{1}{\sqrt{\psi(t)\psi(t')}}\left[\psi\left(t'+\frac{\tau}{2}\right) - 2T\int_0^{\frac{\tau}{2}}{\rm d}y\, f(y)\psi\left(t'+\frac{\tau}{2}-y\right)\right]. \end{aligned} \tag{A.101}$$

It is possible to use this expression for calculating the asymptotic forms of autocorrelation. One then recovers the same asymptotic results that have already been reported, with the exception a discrepancy in the critical dynamics for $d > \overline{d}$, which will not display any ageing behaviour. Equation (A.101), however, involves strongly varying terms, which may even change sign, and whose asymptotic behaviour may turn out to be much more difficult to analyse.

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